Section I

NESA Number:

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. What is the domain and range of $y = 2\sin^{-1}\dfrac{2x}{5}$?

(A) Domain: $-\dfrac{5}{2} \le x \le \dfrac{5}{2}$, Range: $-\dfrac{\pi}{4} \le y \le \dfrac{\pi}{4}$.

(B) Domain: $-\dfrac{2}{5} \le x \le \dfrac{2}{5}$, Range: $-\pi \le y \le \pi$.

(C) Domain: $-\dfrac{5}{2} \le x \le \dfrac{5}{2}$, Range: $-\pi \le y \le \pi$.

(D) Domain: $-\dfrac{2}{5} \le x \le \dfrac{2}{5}$, Range: $-\dfrac{\pi}{4} \le y \le \dfrac{\pi}{4}$

2. Given $\overrightarrow{OA} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{AB} = 4\underset{\sim}{i} - \underset{\sim}{j}$, which is the correct value for $\overrightarrow{OB}$?

(A) $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ (B) $\begin{pmatrix} -6 \\ 2 \end{pmatrix}$

(C) $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$ (D) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

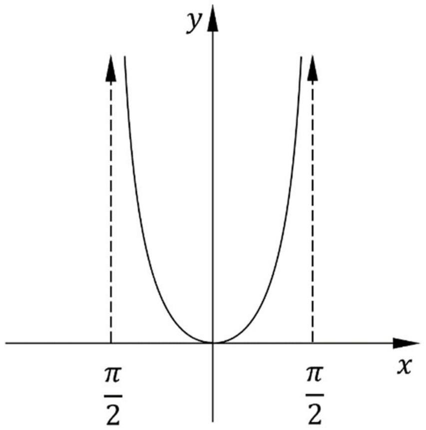
3. What is the remainder when $P(x) = x^3 - 2x + 3$ is divided by $(2x-1)$

(A) $3\dfrac{7}{8}$ (B) $2\dfrac{1}{8}$

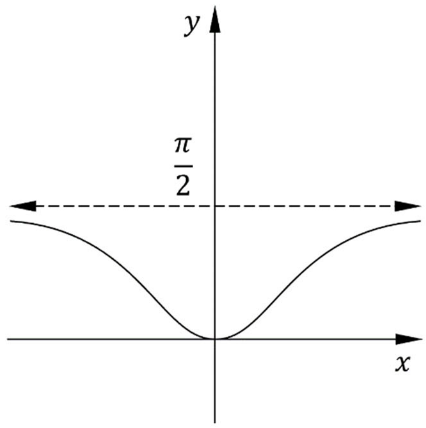
(C) 2 (D) 4

4. Which of the following graphs best shows $y = \tan^{-1}(x^2)$?

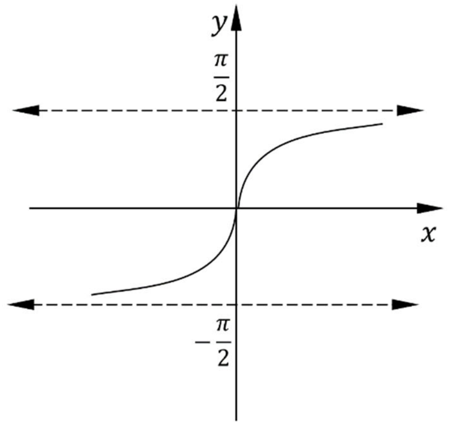
(a)



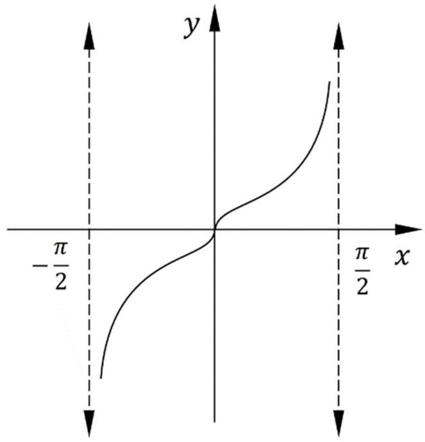
(B)



(C)



(D)



5. Consider the differential equation $\dfrac{dy}{dx} = 4xy$.

Which of the following is the family of solutions to the equation.

(A) $y = Ae^{2x^2}$

(B) $y = \ln(2x^2) + c$

(C) $y = 2x^2 \ln|y| + c$

(D) $y = 4x \ln|y| + c$

6. The cartesian equation of the curve with the parametric equations $x = 2e^t$ and $y = \cos(1 + e^{3t})$ for $0 \le t \le \dfrac{3}{4}$ is given by:

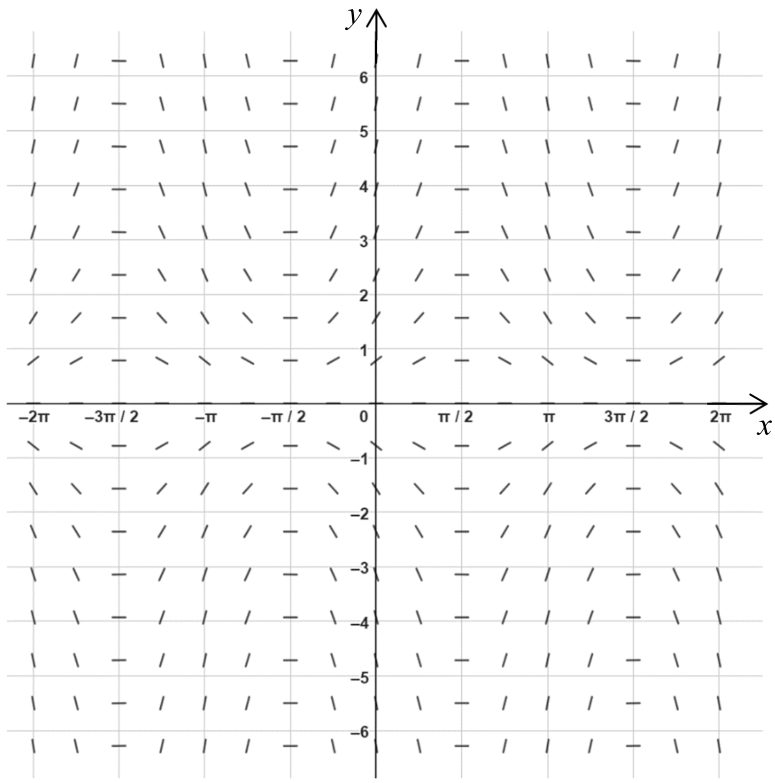
(A) $y = \cos\left(1 + \dfrac{e^3}{8}x\right)$

(B) $y = \cos\left(1 + \dfrac{x}{2}\right)$

(C) $y = \cos\left(1 + \dfrac{x}{2} + e^3\right)$

(D) $y = \cos\left(1 + \dfrac{x^3}{8}\right)$

7. Which differential equation is shown in the slopefield below?



(A) $y' = y\cos x$ (B) $y' = y\sin x$

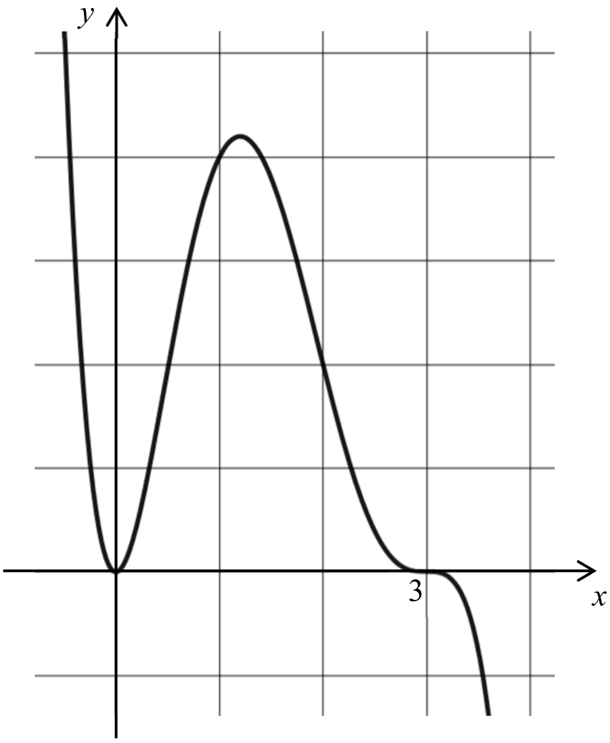
(C) $y' = x\cos y$ (D) $y' = x\sin y$

8. What is the value of k such that $\int_0^k \frac{1}{\sqrt{4-9x^2}}\,dx = \frac{\pi}{18}$

(A) -3 (B) $\frac{1}{3}$

(C) $-\frac{1}{3}$ (D) 3

9. Which of the following could be the polynomial $y = P(x)$.



(A) $y = x^3(x-3)^2$

(B) $y = x^2(x-3)^3$

(C) $y = -x^3(x-3)^2$

(D) $y = -x^2(x-3)^3$

10. The integral $\int_0^{\frac{\pi}{8}} \cos 6x \cos 2x \, dx$ simplified is equal to:

(A) $\frac{3}{16}$

(B) $\frac{1}{8}$

(C) 0

(D) $\frac{1}{16}$

Section II

60 marks
Attempt Questions 1 – 4
Allow about 1 hour and 45 minutes for this section
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 1 (16 marks) - Start your work in Question 1 Answer Booklet

(a) If $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 4\underset{\sim}{j}$, calculate:

(i) $\underset{\sim}{b} - \underset{\sim}{a}$ **1**

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **1**

(b) Differentiate $y = \frac{1}{3}\tan^{-1} 3x$. **2**

(c) Find $\int \frac{1}{x^2 + 2x + 5}\, dx$ **2**

(d) Evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\, dx$ using the substitution $x = \sin^2\theta$. **3**

(e) (i) If the polynomials $P(x) = 2x^3 + mx^2 + 2x - 3$ and $Q(x) = x^2 + nx - 3$ have the same remainder when divided by $x + 2$, write an expression for m in terms of n. **2**

(ii) Given that $(x - 3)$ is a factor of $Q(x)$, find the value of m and n. **2**

(f) Find the exact value of $\cos\frac{\pi}{8}$ giving your answer in simplest form. **3**

End of Question 1.

Question 2 (16 marks) - Start your work in Question 2 Answer Booklet

(a) Prove, by Mathematical Induction, that $5^{2n+1}+2^{2n+1}$ is divisible by 7 for all integers $n\geq 1$. **3**

(b) In a 16 member soccer squad 12 are right-handed while 4 are left-handed. If 11 members are to be selected as the starting line up (players participating at the start of a game), in how many ways will there be at least three left-handed players in the starting line up? **2**

(c) The functions f and g are defined by $f(x)=\sqrt{4-x^2}$ and $g(x)=x-1$.

(i) Show that the domain of $f(g(x))$ is $-1\leq x\leq 3$. **1**

(ii) Hence state the range of the function $f(g(x))$. **1**

(iii) What is the largest domain which includes the point (3 , 0) over which $f(g(x))$ has an inverse function? **1**

(iv) Hence find $h(x)$, the inverse function of the composite function $f(g(x))$, stating its domain and range. **3**

(v) Sketch the graph of $y=h(x)$. **2**

(d) Show that $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$ **3**

End of Question 2.

Question 3 (15 marks) - Start your work in Question 3 Answer Booklet

(a) For vectors $\underset{\sim}{u} = 3\underset{\sim}{i} + b\underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} - 3\underset{\sim}{j}$

(i) Write an expression for the projection of vector u onto vector v. **2**

(ii) Given that the length of this projection is 3 units, find the value of b. **2**

(b) Find in the form $y = f(x)$ the solution of the differential equation $y' = \dfrac{2e^{-\frac{1}{2}y}}{\cos^2 x}$,

given that $y = \ln 3$ when $x = \dfrac{\pi}{3}$. **3**

(c) Newton's law of cooling states that when an object at temperature $T°$ C is placed in an environment at temperature $A°$ C, the rate of temperature loss is given by the equation:

$\dfrac{dT}{dt} = -k(T - A)$ where t is the time in minutes and k is a positive constant.

(i) Use differential equations to show that $T = A + Be^{-kt}$ is a solution to the above equation. **1**

(ii) A cup of tea with initial temperature of 90˚C is placed in a room in which the surrounding temperature is maintained at 25˚C. After 25 minutes, the temperature of the cup of tea is 45˚C. How long will it take for the it's temperature to reduce to 30˚C? Answer correct to the nearest minute. **3**

(d) (i) Ten friends are going to be divided into groups of 5, 3 and 2 members as part of a competition. In how many ways can the groups be formed? **2**

(ii) If n friends are divided into groups of made up of c, k, and r members

where $c + r + k = n$ and $c > k > r$. Explain why

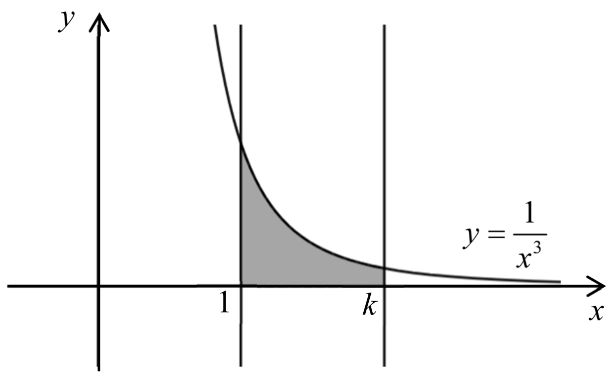
$$\binom{n}{n-k-r}\binom{k+r}{r} = \binom{n}{r}\binom{n-r}{k}.$$ **2**

End of Question 3.

Question 4 (13 marks) - Start your work in Question 4 Answer Booklet

(a) A spherical ball is expanding so that its volume is increasing at the constant rate of 10 mm^3 per seconds. What is the rate of increase of the radius when the surface area is 400 mm^2? **2**

(b) The graph of $y = \dfrac{1}{x^3}$ $\{x > 0\}$ is shown below. The shaded area is rotated about the y-axis.

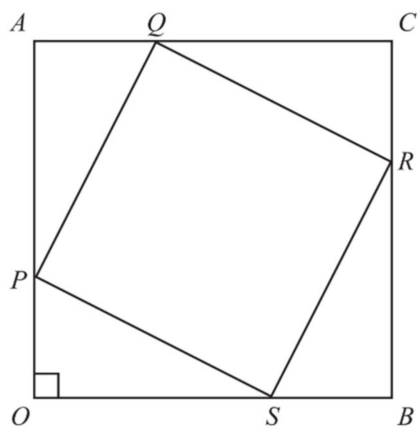


(i) Show that the generated volume in terms of k is $V = \left(2\pi - \dfrac{2\pi}{k}\right)$ units^3. **4**

(ii) Explain what happens to the volume as $k \to \infty$. **1**

(iii) If the volume of the solid form is $\dfrac{3\pi}{2}$ units^3, find the value of k. **1**

(c) Consider the square $OACB$ where point O is the origin. Let the position vector of points A and B be defined as $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively i.e. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



Let points P, Q, R and S be defined so that $\overrightarrow{OP} = k\underset{\sim}{a}$, $\overrightarrow{AQ} = k\underset{\sim}{b}$, $\overrightarrow{RC} = k\underset{\sim}{a}$ and $\overrightarrow{SB} = k\underset{\sim}{b}$ where $0 \le k \le 1$. This means points P, Q, R and S are positioned along their respective sides in equal proportions.

Use vector methods to prove that the size of $\angle PQR = 90°$. **5**

End of Examination.

Section I - Solutions

NESA Number:

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. What is the domain and range of $y = 2\sin^{-1}\dfrac{2x}{5}$?

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(D) Domain: $-\dfrac{2}{5} \le x \le \dfrac{2}{5}$, Range: $-\dfrac{\pi}{4} \le y \le \dfrac{\pi}{4}$

2. Given $\overrightarrow{OA} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{AB} = 4\underset{\sim}{i} - \underset{\sim}{j}$, which is the correct value for $\overrightarrow{OB}$?

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4. Which of the following graphs best shows $y = (x^2)$?

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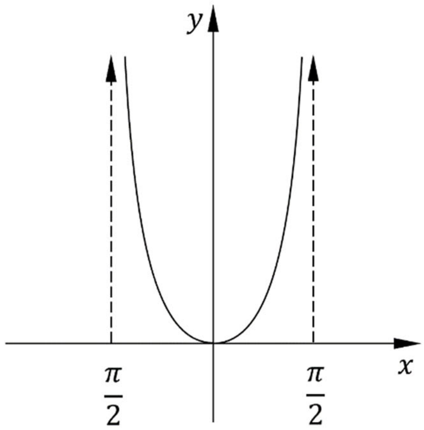
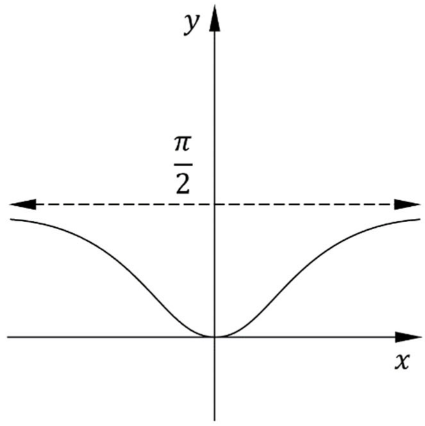
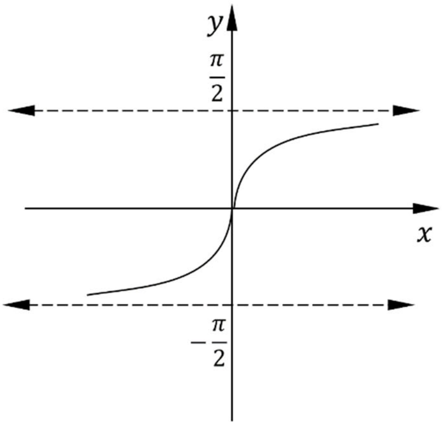


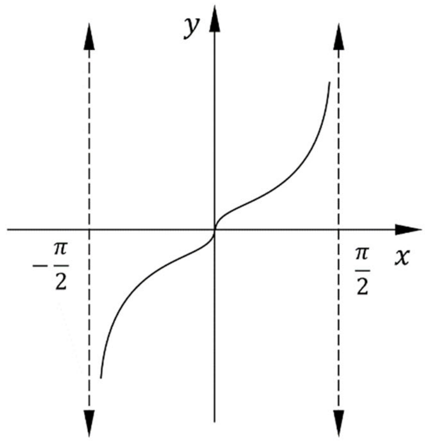
(B)



(C)



(D)



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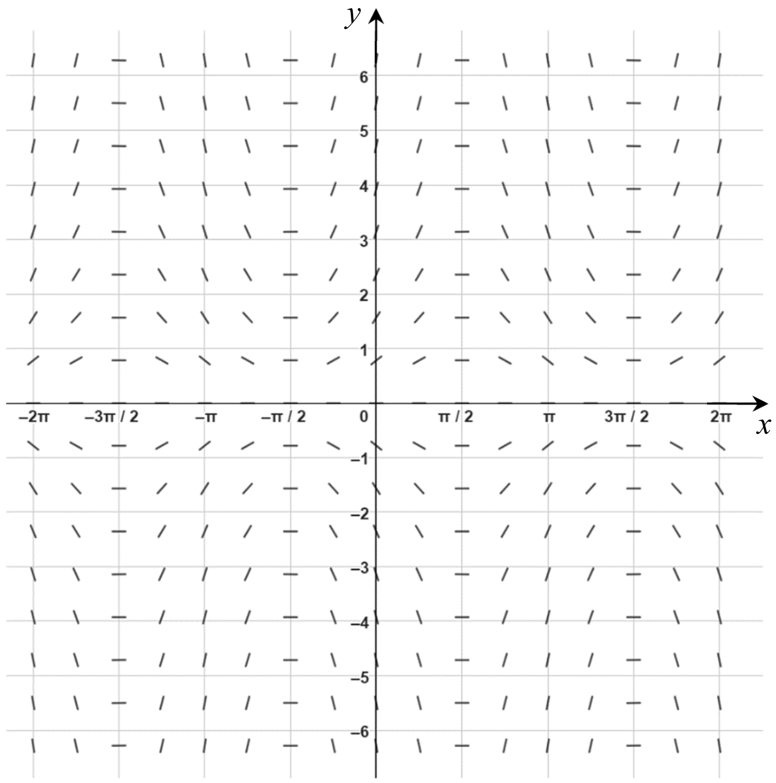

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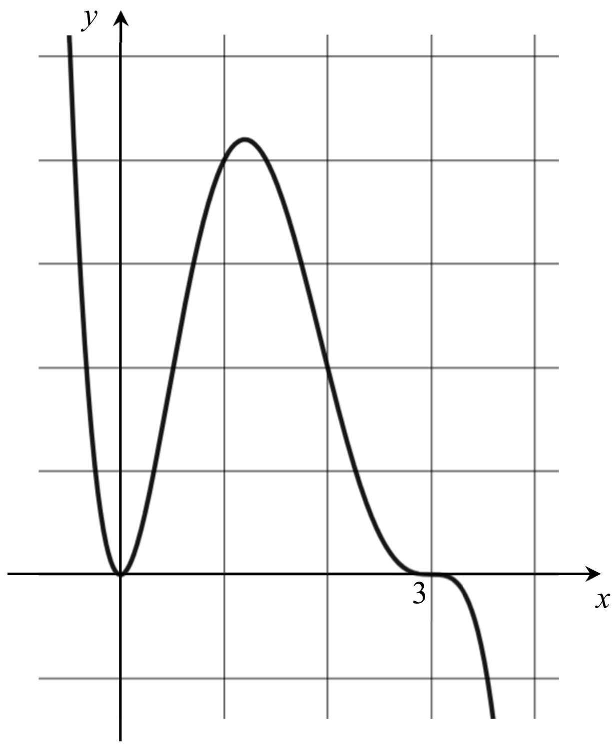
(C) $-\frac{1}{3}$ (D) 3

9. Which of the following could be the polynomial $y = P(x)$.



(A) $y = x^3(x-3)^2$

(B) $y = x^2(x-3)^3$

(C) $y = -x^3(x-3)^2$

(D) $y = -x^2(x-3)^3$

10. The integral $\int_0^{\frac{\pi}{8}} \cos 6x \cos 2x \, dx$ simplified is equal to:

(A) $\frac{3}{16}$

(B) $\frac{1}{8}$

(C) 0

(D) $\frac{1}{16}$

Section II

Question 1 (16marks) - Start your work in Question 1 Answer Booklet

(a) If $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 4\underset{\sim}{j}$, calculate:

(i) $\underset{\sim}{b} - \underset{\sim}{a}$ **1**

$\underset{\sim}{b} - \underset{\sim}{a} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $= -4\underset{\sim}{i} + 6\underset{\sim}{j}$	1 – for answer Marker's Comments: Answered well.

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **1**

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3)(-1) + (-2)(4)$ $= -3 - 8$ $= -11$	1 – for answer Marker's Comments: Answered well.

(b) Differentiate $y = \frac{1}{3}\tan^{-1} 3x$. **2**

$y = \frac{1}{3}\tan^{-1} 3x$ $y' = \frac{1}{3} \frac{3}{1+9x^2}$ $= \frac{1}{1+9x^2}$	1 – for differentiation 1 – for simplified answer Marker's Comments: Answered well.

(c) Find $\int \frac{1}{x^2 + 2x + 5} dx$ **2**

$\int \frac{1}{x^2+2x+5} dx = \int \frac{1}{(x+1)^2 + 2^2} dx$ $= \frac{1}{2}\tan^{-1} \frac{x+1}{2} + C$	1 – for rearrangement 1 – answer Marker's Comments: Most students who new the strategy got full marks. Those who did not used either logs or attempted to incorrectly split the fraction, both leading to incorrect answers.

(d) Evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\, dx$ using the substitution $x = \sin^2\theta$. **3**

$x = \sin^2\theta$ $\frac{dx}{d\theta} = 2\sin\theta\cos\theta$ when $x = 0$ $\theta = 0$ $x = \frac{1}{2}$ $\theta = \frac{\pi}{4}$ $\therefore \int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$ $= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$ $= \int_0^{\frac{\pi}{4}} 2\sin^2\theta\, d\theta$ $[\cos 2\theta = 1 - 2\sin^2\theta]$ $= \int_0^{\frac{\pi}{4}} (1 - \cos 2\theta)\, d\theta$ $= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\sin 0\right)$ $= \frac{\pi}{4} - \frac{1}{2}$	1 – rewriting the integral in terms of theta 1 – integrating 1 – final exact answer. Marker's comments: Many students struggled with this question. Most students made the question more complicated by not using the basic chain rule when differentiating $x = \sin^2\theta$.

(e) (i) If the polynomials $P(x) = 2x^3 + mx^2 + 2x - 3$ and $Q(x) = x^2 + nx - 3$ have the same remainder when divided by $x + 2$, write an expression for m in terms of n. **2**

$P(-2) = -16 + 4m - 4 - 3$ $= 4m - 23$ $Q(-2) = 4 - 2n - 3$ $= 1 - 2n$ Now $P(-2) = Q(-2)$ $\Rightarrow 4m - 23 = 1 - 2n$ $4m = 24 - 2n$ $m = 6 - \frac{n}{2}$	1 – for equating the remainders 1 – for the expression Marker's Comments: Answered well. Students who used the remainder theorem were mostly successful. Those who used long division to find the remainders, made mistakes, leading to incorrect answers.

(ii) Given that $(x-3)$ is a factor of $Q(x)$, find the value of m and n. **2**

$Q(3) = 9 + 3n - 3$ $\therefore \quad 3n + 6 = 0$ $n = -2$ $m = 8$	1 – for value of n 1 – value of m Marker's Comments: Answered well.

(f) Find the exact value of $\cos\dfrac{\pi}{8}$ giving your answer in simplest form. **3**

$\cos 2\theta = 2\cos^2\theta - 1$ $\Rightarrow \cos^2\theta = \frac{1}{2}[\cos 2\theta + 1]$ $\therefore \cos^2\frac{\pi}{8} = \frac{1}{2}\left(\cos\frac{\pi}{4} + 1\right)$ $= \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right)$ $= \frac{1}{2}\left(\frac{\sqrt{2}}{2} + 1\right)$ $= \frac{\sqrt{2}+2}{4}$ $\therefore \cos\frac{\pi}{8} = \frac{(\sqrt{2}+2)^{1/2}}{2}, \left(\cos\frac{\pi}{8} > 0\right)$ (alt answer: $\cos\frac{\pi}{8} = \left(\frac{1}{\sqrt{2}} + 1\right)^{1/2}$)	1 – for establishing the initial relationship 1 – for correct expression 1 – simplified answer with reason for ignoring the negative case. Marker's comments: Generally answered well. Most students made a start achieving one mark. But a number of students ignored the negative case when finding the square root, loosing the chance toe explain why the positive case is the correct answer.

End of Question 1.

Question 2 (16 marks) - Start your work in Question 2 Answer Booklet

(a) Prove, by Mathematical Induction, that $5^{2n+1}+2^{2n+1}$ is divisible by 7 for all integers $n \geq 1$. **3**

Step 1: Prove for $n=1$

Proof: $5^{2+1}+2^{2+1} = 5^3+2^3$

$= 133$

$= 7 \times 19$

$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$ where k is a positive integer

i.e. $5^{2k+1}+2^{2k+1} = 7M$ where M is an integer

Prove true for $n=k+1$

i.e. Prove $5^{2k+3}+2^{2k+3} = 7N$ where N is an integer

Proof:

$\text{LHS} = 5^2 \times 5^{2k+1} + 2^{2k+3}$

$= 5^2\left(7M - 2^{2k+1}\right) + 4 \times 2^{2k+1}$ (by assumption)

$= 7 \times 25M + (4-25)\,2^{2k+1}$

$= 7 \times 25M - 7 \times 3\left(2^{2k+1}\right)$

$= 7\left(25M - 3 \times 2^{2k+1}\right)$

$= 7N$ where $N = 25M - 3 \times 2^{2k+1}$

$= \text{RHS}$

Step 3: Since true for $n=1$, by step 2 and mathematical induction, true for all positive integer n.

1 – proof for $n = 1$
1 – correct establishment of the assumption
1 – for the proof for $n = k+1$ case

Marker's comments:
Generally answered well but marks were deducted if k was not defined as a positive integer.

(b) In a 16 member soccer squad 12 are right-handed while 4 are left-handed. If 11 members are to be selected as the starting line up (players participating at the start of a game), in how many ways will there be at least three left-handed players in the starting line up? **2**

$\binom{4}{3}\binom{12}{8} + \binom{4}{4}\binom{12}{7} = 1980 + 792$ $= 2772$	1 – for first case 1 – for final answer
	Marker's Comments: Generally answered well

(c) The functions f and g are defined by $f(x)=\sqrt{4-x^2}$ and $g(x)=x-1$.

(i) Show that the domain of $f(g(x))$ is $-1 \le x \le 3$. **1**

$f(g(x)) = \sqrt{4-(x-1)^2}$ $4-(x-1)^2 \geqslant 0$ $(x-1)^2 \leqslant 4$ $-2 \leqslant x-1 \leqslant 2$ $D: \{-1 \leqslant x \leqslant 3\}$ ***Alternative approach:*** Domain of $f(x)$: $-2 \le x \le 2$ Domain of $f(g(x))$: $-2 \le x-1 \le 2$ $-1 \le x \le 3$	1 – steps towards finding the domain
	Marker's comments: Most students answered this question well, however, as this is a show question, students should include all necessary steps, including a stating that $4-(x-1)^2 \ge 0$

(ii) Hence state the range of the function $f(g(x))$. **1**

$R: \{0 \leqslant y \leqslant 2\}$	1 – range
	Marker's comments: Generally answered well.

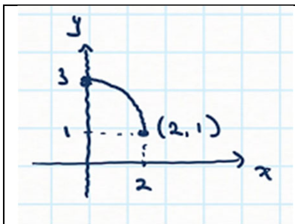
(iii) What is the largest domain which includes the point (3 , 0) over which $f(g(x))$ has an inverse function? **1**

$D: \{1 \leqslant x \leqslant 3\}$	1 – domain
	Marker's comments: Generally answered well.

(iv) Hence find $h(x)$, the inverse function of the composite function $f(g(x))$, stating its domain and range. **3**

let $x = \sqrt{4-(y-1)^2}$ $x^2 = 4 - (y-1)^2$ $(y-1)^2 = 4 - x^2$ $y - 1 = \pm\sqrt{4-x^2}$ $y = 1 \pm \sqrt{4-x^2}$ D: $\{0 \leqslant x \leqslant 2\}$ $\therefore h(x) = 1 + \sqrt{4-x^2}$ R: $\{1 \leqslant y \leqslant 3\}$	1 – correct domain & range 1 – expression for y 1 – establishing the positive case as the function $h(x)$ **Marker's comments:** Students had difficulty identifying the correct domain and range. Many did not provide a rationale for taking the positive square root int their final answer. Had this been a show question a mark would have been deducted.

(v) Sketch the graph of $y = h(x)$. **2**

	1 – correct section of circle 1 – indicating the end points **Marker's comments:** Not answered very well due to the difficulties students had when answering part (iv)

(d) Show that $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ 3

$\tan^{-1} 1 = \frac{\pi}{4}$ let $\alpha = \tan^{-1} 2$; $\beta = \tan^{-1} 3$ then $\tan\alpha = 2$; $\tan\beta = 3$ $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ $= \frac{2+3}{1-6}$ $= \frac{5}{-5}$ $= -1$ $\therefore \alpha+\beta = \tan^{-1}(-1)$ $= \frac{3\pi}{4}$ $\therefore \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \frac{\pi}{4} + \frac{3\pi}{4}$ $= \pi$	1 - $\tan^{-1}1 = \frac{\pi}{4}$ 1 – establishing $\tan(\alpha+\beta)$ 1 – demonstrating LHS = π **Marker's comments:** Not answered very well. Students did not readily recognise that $1 = \frac{\pi}{4}$ and as such had difficulty progressing with the solution. A few arrived at the correct proof but through very complicated approaches. Some students used the calculator to evaluate 2 and 3 which gave approximate values and consequently a mark was deducted. Others, incorrectly, tried applying the compound angle formula using three terms instead of two.

End of Question 2.

Question 3 (15 marks) - Start your work in Question 3 Answer Booklet

(a) For vectors $\underset{\sim}{u} = 3\underset{\sim}{i} + b\underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} - 3\underset{\sim}{j}$

(i) Write an expression for the projection of vector *u* onto vector *v*. **2**

$\text{Proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{v}\rvert^2}\underset{\sim}{v}$ $\underset{\sim}{u}\cdot\underset{\sim}{v} = (3)(-1) + (b)(-3)$ $= -3 - 3b$ $\lvert\underset{\sim}{v}\rvert^2 = (-1)^2 + (-3)^2$ $= 10$ $\therefore \text{Proj}_{\underset{\sim}{v}} \underset{\sim}{u} = -\frac{3}{10}(1+b)(-\underset{\sim}{i} - 3\underset{\sim}{j})$ $= \frac{3(1+b)}{10}\underset{\sim}{i} + \frac{9(1+b)}{10}\underset{\sim}{j}$	1 – for $\underset{\sim}{u}\cdot\underset{\sim}{v}$ 1 – solution Marker's comment: Answered well.

(ii) Given that the length of this projection is 3 units, find the value of *b*. **2**

$\lvert\text{Proj}_{\underset{\sim}{v}} \underset{\sim}{u}\rvert = \pm\frac{(1+b)}{10}\sqrt{9+81}$ $= \pm\frac{3\sqrt{10}}{10}(1+b)$ $\therefore \pm\frac{3\sqrt{10}}{10}(1+b) = 3$ $(1+b) = \pm\frac{10}{\sqrt{10}}$ $b = \pm\sqrt{10} - 1$	1 – length of projection in terms of *b* 1 – for answer Marker's comments: Many students forgot to do only the positive answer.

(b) Find in the form $y = f(x)$ the solution of the differential equation $y' = \dfrac{2e^{-\frac{1}{2}y}}{\cos^2 x}$,

given that $y = \ln 3$ when $x = \dfrac{\pi}{3}$. **3**

$\dfrac{dy}{dx} = \dfrac{2e^{-\frac{1}{2}y}}{\cos^2 x}$ $\int \dfrac{1}{2e^{-\frac{1}{2}y}}\,dy = \int \dfrac{1}{\cos^2 x}\,dx$ $\int \frac{1}{2}e^{\frac{1}{2}y}\,dy = \int \sec^2 x\,dx$ $e^{\frac{1}{2}y} = \tan x + C$ $\left.\begin{array}{l} x=\frac{\pi}{3} \\ y=\ln 3\end{array}\right\}$ $e^{\frac{1}{2}\ln 3} = \tan\frac{\pi}{3} + C$ $\sqrt{3} = \sqrt{3} + C$ $\therefore\ C = 0$ $\therefore\ e^{\frac{y}{2}} = \tan x$ $\frac{y}{2} = \ln(\tan x)$ $y = 2\ln(\tan x)$	1 – Establishing the integrals 1 – correctly integrating both integrals 1 – correct final expression for y Marker's comments: Setting of working was a problem, with many forgetting to include integral signs, or unable to provide the first integral statement at all . The integral of the exponential was done badly by many.

(c) Newton's law of cooling states that when an object at temperature $T°$ C is placed in an environment at temperature $A°$C, the rate of temperature loss is given by the equation:

$\dfrac{dT}{dt} = -k(T - A)$ where t is the time in minutes and k is a positive constant.

(i) Use differential equations to show that $T = A + Be^{-kt}$ is a solution to the above equation. **1**

$\dfrac{dT}{dt} = -k\,(T-A)$ $\int \dfrac{1}{T-A}\,dT = \int -k\,dt$ $\ln\lvert T-A\rvert = -kt + c$ $T - A = e^{-kt}\cdot e^{c}$ $T = A + Be^{-kt}$ where $B = e^{c}$	1 – demonstrate through integration Many students did not understand to integrate first , rather than start with the solution.

(ii) A cup of tea with initial temperature of 90˚C is placed in a room in which the surrounding temperature is maintained at 25˚C. After 25 minutes, the temperature of the cup of tea is 45˚C. How long will it take for the it's temperature to reduce to 30˚C? Answer correct to the nearest minute. **3**

$\left.\begin{array}{l} t=0 \\ T=90^\circ \\ A=25^\circ \end{array}\right\\}$ $90 = 25 + Be^{0} \Rightarrow B = 65$ $\therefore T = 25 + 65e^{-kt}$ $\left.\begin{array}{l} t=25 \\ T=45 \end{array}\right\\}$ $45 = 25 + 65e^{-25k}$ $e^{-25k} = \frac{4}{13} \Rightarrow k = -\frac{1}{25}\ln\frac{4}{13}$ $\therefore T = 25 + 65e^{\frac{t}{25}\ln\frac{4}{13}}$ $T=30 \Rightarrow 30 = 25 + 65e^{\frac{t}{25}\ln\frac{4}{13}}$ $e^{\frac{t}{25}\ln\frac{4}{13}} = \frac{5}{65}$ $\frac{t}{25}\ln\frac{4}{13} = \ln\frac{1}{13}$ $t = 25\frac{\ln 1/13}{\ln 4/13}$ $\doteqdot 54$ min.	1 – finding B 1 – finding k 1 – for final answer Marker's comments: Answered well.

(d) (i) Ten friends are going to be divided into groups of 5, 3 and 2 members as part of a competition. In how many ways can the groups be formed? **2**

$\binom{10}{5}\binom{5}{3} = 2520$	1 – for one of the combinations 1 – for final answer Marker's comments: Answered well.

(ii) If n friends are divided into groups of made up of c, k, and r members

where $c+r+k=n$ and $c>k>r$. Explain why

$$\binom{n}{n-k-r}\binom{k+r}{r}=\binom{n}{r}\binom{n-r}{k}.$$ **2**

A group of n friends are being divided into 3 groups of numbers c, k and r. Considering LHS, if we choose the 1st 'c' friends then k+r are left (note: c = n-k-r). Then from k+r we choose k, leaving 'r' friends. Now considering RHS, we can at first select 'r' friends, leaving 'n-r' friends. Then from 'n-r' we choose k friends leaving c = n-k-r friends, resulting in the same number of groupings.	1 – for identifying the selection process on the LHS 1 – for similarly working on the RHS Marker's comments: Well done by those who attempted it

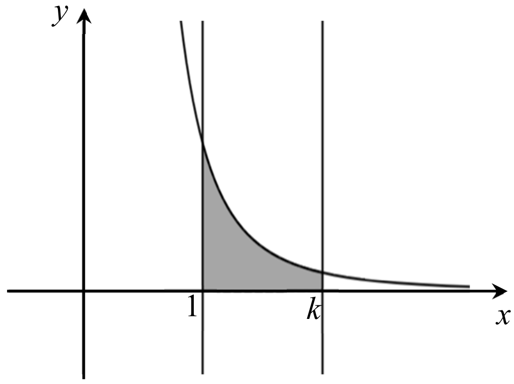
End of Question 3.

Question 4 (13 marks) - Start your work in Question 4 Answer Booklet

(a) A spherical ball is expanding so that its volume is increasing at the constant rate of 10 mm^3 per seconds. What is the rate of increase of the radius when the surface area is 400 mm^2? **2**

<table>
<tr>
<td rowspan="2">$\frac{dV}{dt} = 10\ mm^3/sec$; $\frac{dr}{dt} = ?$ $SA = 400 mm^2$
$V = \frac{4}{3}\pi r^3$ $SA = 4\pi r^2$
$\frac{dV}{dr} = 4\pi r^2$ $= 400$
$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$
$= \frac{1}{4\pi r^2} \cdot 10$
$\therefore \frac{dr}{dt} = \frac{10}{400}$
$= \frac{1}{40}$ mm/sec</td>
<td>1 – for writing the expression for dr/dt
1 – for final answer</td>
</tr>
<tr>
<td>Marker's comments:
Well done</td>
</tr>
</table>

(b) The graph of $y = \dfrac{1}{x^3}$ $\{x > 0\}$ is shown below. The shaded area is rotated about the y-axis.



(i) Show that the generated volume in terms of k is $V = \left(2\pi - \dfrac{2\pi}{k}\right)$ units^3. **4**

$y = \frac{1}{x^3} \Rightarrow x^3 = \frac{1}{y}$ $x^2 = y^{-\frac{2}{3}}$ when $x = 1 \Rightarrow y = 1$ $x = k \Rightarrow y = \frac{1}{k^3}$ $V = \pi\int_a^b x^2\,dy + \pi r_1^2 h - \pi r_2^2 h$ $= \pi\int_{1/k^3}^{1} y^{-\frac{2}{3}}\,dy + \pi(k)^2\left(\frac{1}{k^3}\right) - \pi(1)^2(1)$ $= \pi\left[3y^{\frac{1}{3}}\right]_{1/k^3}^{1} + \frac{\pi}{k} - \pi$ $= 3\pi\left[1 - \left(\frac{1}{k^3}\right)^{\frac{1}{3}}\right] + \frac{\pi}{k} - \pi$ $= 3\pi - \frac{3\pi}{k} + \frac{\pi}{k} - \pi$ $= \left(2\pi - \frac{2\pi}{k}\right)$ units^3	1 – Integral for the curve with correct limits 1 – Volume for both cylinders 1 – correct integration 1 – for correct simplification Marker's comments: Poorly done . Most of the students couldn't figure out the correct area that is being rotated around y axis .

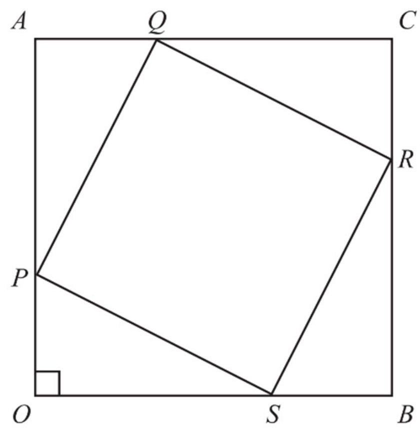
(ii) Explain what happens to the volume as $k \to \infty$. **1**

As $k \to \infty$, $\frac{2\pi}{k} \to 0 \Rightarrow V = 2\pi$ units^3	1 – for the answer Marker's comments: well done

(iii) If the volume of the solid form is $\dfrac{3\pi}{2}$ units^3, find the value of k. **1**

$2\pi - \frac{2\pi}{k} = \frac{3\pi}{2}$ $\frac{2\pi}{k} = \frac{\pi}{2}$ $k = 4$ $\therefore k = 6$	1 – for correct answer Marker's comments: very well done

(c) Consider the square $OACB$ where point O is the origin. Let the position vector of points A and B be defined as $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively i.e. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



Let points P, Q, R and S be defined so that $\overrightarrow{OP} = k\underset{\sim}{a}$, $\overrightarrow{AQ} = k\underset{\sim}{b}$, $\overrightarrow{RC} = k\underset{\sim}{a}$ and $\overrightarrow{SB} = k\underset{\sim}{b}$ where $0 \le k \le 1$. This means points P, Q, R and S are positioned along their respective sides in equal proportions.

Use vector methods to prove that the size of $\angle PQR = 90°$. **5**

$\overrightarrow{PA} = (1-k)\underset{\sim}{a}$; $\overrightarrow{AQ} = k\underset{\sim}{b}$ $\overrightarrow{QC} = (1-k)\underset{\sim}{b}$; $\overrightarrow{CR} = -k\underset{\sim}{a}$ $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$; $\overrightarrow{QR} = \overrightarrow{QC} + \overrightarrow{CR}$ $= (1-k)\underset{\sim}{a} + k\underset{\sim}{b}$; $= (1-k)\underset{\sim}{b} - k\underset{\sim}{a}$ $\overrightarrow{PQ}\cdot\overrightarrow{QR} = [(1-k)\underset{\sim}{a} + k\underset{\sim}{b}]\cdot[(1-k)\underset{\sim}{b} - k\underset{\sim}{a}]$ $= (1-k)^2\underset{\sim}{a}\cdot\underset{\sim}{b} - k^2\underset{\sim}{a}\cdot\underset{\sim}{b}$ Now $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ (OACB is a square) $\therefore \overrightarrow{PQ}\cdot\overrightarrow{QR} = 0$ $\therefore \angle PQR = 90°$	1 – Expression for $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ 1 – Expression for $\overrightarrow{QR}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ 1 – Using the dot product 1 – Simplifying the dot product 1 – $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ Marker's comments: well done . Some students struggled in using the properties of square to the dot product of vector PQ and vector QR .

End of Examination.